Inequality and Macroeconomics: Facts and Theories

Lecture 2. Neoclassical macro models of inequality: theory

Fabrizio Perri
Minneapolis FED and CEPR

KU Leuven
May 2024

The HA building block in HANK

- Large number of agents
- Each facing stochastic income fluctuations (and potential shocks to return to accumulated wealth)
- Each optimally choosing consumption and savings
- General equilibrium (wages and interest rates are endogenous)

A Framework Without Aggregate Uncertainty

- Continuum of measure 1 of individuals, each facing an income fluctuation problem
- Stochastic labor endowment process $\{y_{it}\}_{t=0}^{\infty}$
- Labor endowment process follows stationary Markov process with transitions $\pi(y'|y)$
- Labor income: $w_t y_{it}$
- Law of large numbers: $\pi(y'|y)$ is also the deterministic fraction of the population that has this particular transition.

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- Law of large numbers: $\pi(y'|y)$ is also the deterministic fraction of the population that has this particular transition.
- Stationary distribution associated with π, denoted by Π, assumed to be unique.
- At period 0 income of all agents, y_0, is given. Population distribution given by Π.
- Total labor endowment in the economy at each point of time

$$\bar{L} = \sum_{y} y \Pi(y)$$

Examples for y (log of income) process

- y is a Markov Chain with N states with transition probability denoted by a nxn matrix Q. In this case the stationary distribution is the $nx1$ eigen-vector of Q associated with the unit eigen-value of Q. If Q is stochastic and has all non zero elements the stationary distribution exists and is unique.
- y is an AR process $y_t = \bar{y} + \rho y_{t-1} + \epsilon_t$, where $\epsilon \to N(0, \sigma^2)$. In this case the stationary distribution is given by $N(\frac{\bar{y}}{1-\rho}, \frac{1}{1-\rho^2}\sigma^2)$
- y is the sum of three components. A fixed effect z_i, a persistent component $y_{it} = \rho y_{it-1} + \epsilon_{it}$ and a purely transitory component $x_{it} = \eta_{it}$ (better fit of observed houshold income histories)

Substantial idiosyncratic uncertainty, but no aggregate uncertainty. Look for stationary equilibrium with constant w and r

Preferences and Budget Constraints

- Preferences
$$u(c) = E_0 \sum_{t=0}^{\infty} \beta^t U(c_t)$$
with $0 < \beta < 1$
- Budget constraint
$$c_t + a_{t+1} = w_t y_t + (1 + r_t) a_t$$
- Borrowing constraint $a_{t+1} \geq -\bar{a}$
- Note that labor is supplied inelastically, saving is in a single asset and is risk free (housing? stock market?)

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- Initial conditions of agent (a_0, y_0) with initial population measure $\lambda_0(a_0, y_0)$
- Allocation: $\{c_t(a_0, y^t), a_{t+1}(a_0, y^t)\}$

Labor and Capital Demand

- Solving the household problem above yields an aggregate labor supply and an aggregate capital supply.
- In equilibrium these will be equated to aggregate labor and capital demand $L(w)$ and $K(r)$

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- Examples:
- Pure Credit Economy (Huggett, 1993). Firms have a linear technology $Y = L$ so $w = 1$, and when $w = 1$ labor demand always equal supply (equivalent to a backyard technology). Capital demand is 0.
- Production Economy (Aiyagari, 1994) Firms use a CRS technology $Y = F(K, L) + (1 - \delta)K$ and capital and labor demand are implicitly defined by

$$\begin{aligned} r &= F_k(K, L) - \delta \\ w &= F_L(K, L) \end{aligned}$$

Recursive Equilibrium

- Individual state (a, y)
- Aggregate state variable: $\lambda(a, y)$ in the sense that prices might depend on the distribution of resources in the economy
- $A = [-\bar{a}, \infty)$: set of possible asset holdings
- Y : set of possible labor endowment realizations
- Let the state space be $S = A \times Y$ and all possible subsets of the state space $\mathcal{B}(S)$.
- Let Λ the set of all probability measures on the measurable space $(S, \mathcal{B}(S))$

Household problem in recursive formulation

$$
\begin{aligned}
& v(a, y; \lambda) \\
= \quad & \max_{c \geq 0, a' \geq 0} u(c) + \beta \sum_{y' \in Y} \pi(y'|y) v(a', y'; \lambda')
\end{aligned}
$$

$$
\begin{aligned}
\text{s.t. } c + a' &= w(\lambda) y + (1 + r(\lambda)) a \\
\lambda' &= H(\lambda)
\end{aligned}
$$

- Function $H : \Lambda \to \Lambda$ is called the aggregate “law of motion”

Transition Functions

- How can we obtain next period distribution, given this period distribution?
- Define $Q\left((a, y), \mathcal{A} \times \mathcal{Y}\right)$ as the probability that an individual with current state (a, y) transits to the set $\mathcal{A} \times \mathcal{Y}$ next period, formally $Q : Sx\mathcal{B}(S) \rightarrow [0, 1]$, and

$$Q\left((a, \varepsilon), \mathcal{A} \times \mathcal{E}\right) = \sum_{y' \in \mathcal{Y}} I\left\{a'(a, y) \in \mathcal{A}\right\} \pi(y', y)$$

  where I is the indicator function, $a'(a, y)$ is the optimal saving policy and $\pi(y', y)$ is the transition probability function i.e. the probability of having shock y' tomorrow given that the shock today is y.
- Q is our transition function and the associated T^* operator yields

$$\lambda'\left(\mathcal{A} \times \mathcal{Y}\right) = T^*(\lambda) = \int_{A \times Y} Q\left((a, y), \mathcal{A} \times \mathcal{Y}\right) d\lambda\left(a, y\right).$$

Stationary recursive competitive equilibrium

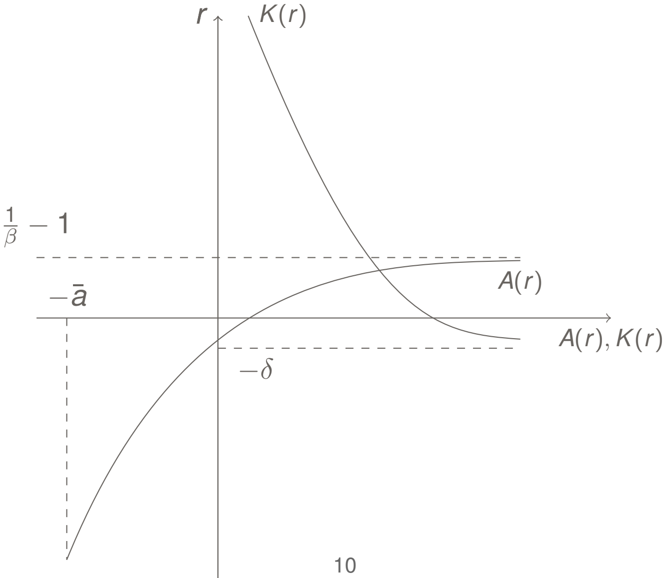
- A **SRCE** is a value function v, policy functions for the household a', and c; demands for savings $K(r)$ and Labor $L(w)$, interest rate r and wages w; and measure $\lambda^* \in \Lambda$ such that:

- given r, w the policy functions a' and c solve the household's problem and v is the associated value function
- the asset market and labor market clears: $K(r) = \int_{A\times E} a'(a,y)d\lambda^*(a,y)$, $L(w) = \int_{A\times E} yd\lambda^*(a,y)$
- for all $(\mathcal{A} \times \mathcal{Y}) \in \mathcal{B}$, the invariant probability measure λ^* satisfies

$$\lambda^* (\mathcal{A} \times \mathcal{Y}) = \int_{A\times Y} Q((a,y), \mathcal{A} \times \mathcal{Y}) d\lambda^* (a,y),$$

where Q is the transition function.

Graphical representation of the equilibrium



The model with exponential utility (Wang, 2003)

- Huggett economy, many agents each with idiosyncratic shock
- No borrowing constraints,
- Wealth in zero net supply
- Note that exponential utility is special as it allows negative consumption.

The Basic Model

- Utility

$$
\begin{aligned}
& E\sum_{t=0}^{\infty}\beta^t u(c_t) \\
u(c) &= -\frac{1}{\gamma}e^{-\gamma c}
\end{aligned}
$$

- Income process

$$
\begin{aligned}
y_t &= \rho_0 + \rho_1 y_{t-1} + \varepsilon_t \\
\varepsilon_t &\to N(0, \sigma) \\
\rho_0 &> 0, 0 < \rho_1 < 1
\end{aligned}
$$

- Budget Constraint

$$
a_{t+1} = (1+r)a_t + y_t - c_t
$$

Permanent Income

Permanent income P_t in terms of today's wealth is

$$P_t = \frac{1}{1+r} E_t \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j y_{t+j}$$

that after some algebra reduces to

$$P_t = \frac{1}{1+r-\rho_1} (y_t + \frac{\rho_0}{r})$$

If $\rho_0 = 0$ we have

$$P_t = \frac{1}{1 + r - \rho_1} y_t$$

that reduces to the familiar

$$P_t = \frac{y_t}{r}$$

in the case in which $\rho_1 = 1$, and

$$P_t = \frac{1}{1 + r} y_t$$

in the case in which $\rho_1 = 0$.

The consumption function

Optimal consumption can be written (using Euler equation and quite a bit of algebra) as

$$\begin{aligned} c_t &= ra_t + \frac{r}{1+r-\rho_1} y_t - \frac{1}{\gamma r}\left(\log(\beta(1+r)) + \log(E(e^{\frac{-\gamma \varepsilon r}{1+r-\rho_1}})\right) \\ c_t &= ra_t + rP_t - \Gamma_1 - \Gamma_2 \end{aligned}$$

Equal to PIH consumption except 2 constants

Saving, 1

Write saving as

$$\begin{aligned} s_t &= ra_t + y_t - c_t \\ &= y_t - \frac{r}{1+r-\rho_1} y_t - \frac{\rho_0}{1+r-\rho_1} \\ &= \frac{y_t(1-\rho_1)}{1+r-\rho_1} - \frac{\rho_0}{1+r-\rho_1} \\ s_t &= (y_t - \bar{y}) \frac{(1-\rho_1)}{1+r-\rho_1} + \Gamma_1 + \Gamma_2 \\ \bar{y} &= \frac{\rho_0}{1-\rho_1} \end{aligned}$$

Saving can be decomposed in three parts

- Rainy days saving: $(y_t - \bar{y})\frac{(1-\rho_1)}{1+r-\rho_1}$
- Precautionary saving $\Gamma_1 = \frac{1}{\gamma r}\left(\log(E(e^{\frac{-\gamma \varepsilon r}{1+r-\rho_1}})\right)$
- Intertemporal substitution saving $\Gamma_2 = \frac{1}{\gamma r}\left(\log(\beta(1+r))\right)$

Saving, 2

Saving for rainy days is always 0 (due to the law of large numbers) so for total saving to be 0 it must be that $\int \Gamma_1 + \int \Gamma_2 = 0$.
Since Γ_1, Γ_2 are constants $\Gamma_1 + \Gamma_2 = 0$
Now if $r = \frac{1}{\beta} - 1, \Gamma_1 + \Gamma_2 > 0$,
if $r = -1, \Gamma_1 + \Gamma_2 = -\infty$
Hence by continuity there exists an $r < \frac{1}{\beta} - 1$ such that $\Gamma_1 + \Gamma_2 = 0$.

- In this economy consumption is exactly like in the PIH. Negative intertemporal substitution saving (because $r < \frac{1}{\beta} - 1$) exactly compensates positive precautionary saving
- Result heavily relies on the fact that total saving is in 0 net supply.

Changes in Consumption

One can derive (easier in in the case of $\rho_1 = 0$ (i.i.d)) an expression for changes in consumption

$$\Delta c_t = \frac{r}{1+r}(y_{t+1} - \rho_0) + \frac{1}{\gamma r}\left(\log(\beta(1+r) + \log E(e^{\frac{-\gamma \varepsilon r}{1+r-\rho_1}})\right)$$

- Consumption is a random walk and the only case in which an equilibrium exists is the one where there is no drift (because there is no drift in income).
- In equilibrium variance of wealth and consumption distribution explode but equilibrium exists because wealth still integrates to 0 (there are just as many agents with infinitely positive wealth as agents with infinitely negative).

Changes in Wealth

$$
\begin{aligned}
a_{t+1} - a_t &= \Delta a_t = y_t + ra_t - c_t \\
&= y_t - rP_t + \Gamma_1 + \Gamma_2 \\
&= \frac{y_t(1-\rho_1)}{1+r-\rho_1} + \Gamma_1 + \Gamma_2
\end{aligned}
$$

shows for example how wealth goes to $+\infty$ if $\beta(1+r) = 1$ and there is even a tiny risk, or how if there is no risk and $\beta(1+r) > 1$